RSK linear operators and the Vershik-Kerov-Logan-Shepp curve

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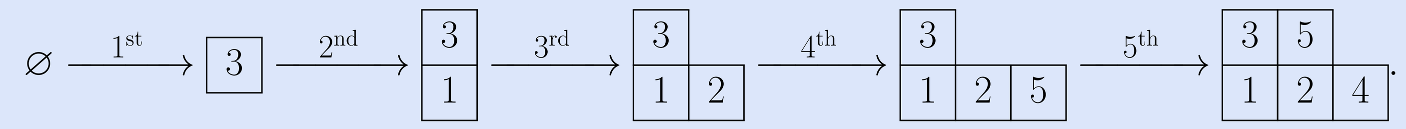
Schensted insertion

Algorithm: From permutations to standard Young tableaux.
Example: Insertion $31254 \in S_5$



Bump types:

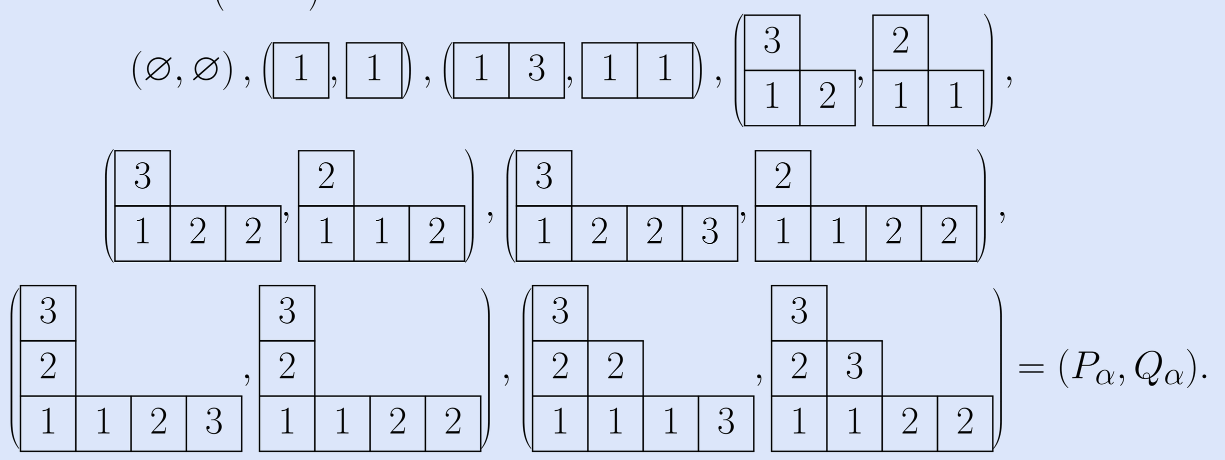
- **Vertical bump**: occurs in the same column (e.g., 2nd step),
- **Lateral bump**: shifts to the left (e.g., 5th step).

Definition: Let V_n be the set of all permutations in S_n with no lateral bumps.

RSK linear operators

Algorithm: RSK correspondence is a bijection between matrices with non-negative integer entries and pairs of semistandard Young tableaux.

Example: From $\alpha = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, we construct $\mathsf{biword}(\alpha) = (1322311 \mid 1122233)$, then insert



Operator: The RSK correspondence defines a linear map on the coordinate ring

$$R_{p,q} := \mathbb{C}[z_{i,j}]_{1\le i\le p, 1\le j\le q} \quad \text{by} \quad z^\alpha \mapsto [P_\alpha, Q_\alpha],$$

where z^α is a monomial (in the monomial basis)

$$z^\alpha := \prod_{1\le i\le p, 1\le j\le q} z_{i,j}^{\alpha_{i,j}},$$

and $[P_\alpha \mid Q_\alpha]$ is a bitableaux (in the bitableaux basis)

$$[P_\alpha \mid Q_\alpha] = \Delta_1\Delta_2\cdots = \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \cdot \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix} \cdot |z_{12}| \cdot |z_{32}|.$$

Invariant subspace: Let $(\sigma, \pi) \in \mathbb{N}^p \times \mathbb{N}^q$ be rows and columns sums, the weight space is

$$R_{\sigma,\pi} = \operatorname{Span}\{z^\alpha \mid \alpha \in \mathsf{Mat}_{\sigma,\pi}(\mathbb{N})\}.$$

Restriction operator: $\mathsf{RSK}_{\sigma,\pi} : R_{\sigma,\pi} \to R_{\sigma,\pi}$ with matrix entries

$$\mathsf{RSK}_{\sigma,\pi}(\beta, \alpha) = \left[z^\beta\right][P_\alpha \mid Q_\alpha].$$

Definition: $C_n = \{\alpha \in \mathsf{Mat}_{1^n,1^n} \mid \mathsf{RSK}_{1^n,1^n}(\alpha,\alpha) = 0\}$.

Results

Theorem 1: Most permutations have a lateral bump in their Schensted insertion, i.e.,

$$\lim_{n\to\infty} \frac{|V_n|}{n!} = 0.$$

Theorem 2 (Stelzer-Yong conjecture): Most diagonal entries of $\mathsf{RSK}_{1^n,1^n}$ vanish, i.e.,

$$\lim_{n\to\infty} \frac{|C_n|}{n!} = 1.$$

Key Lemma: For $\alpha \in \mathsf{Mat}_{1^n,1^n}$, we have

$$\mathsf{RSK}_{1^n,1^n}(\alpha,\alpha) = 0 \iff w_\alpha \notin V_n.$$

Limit shape

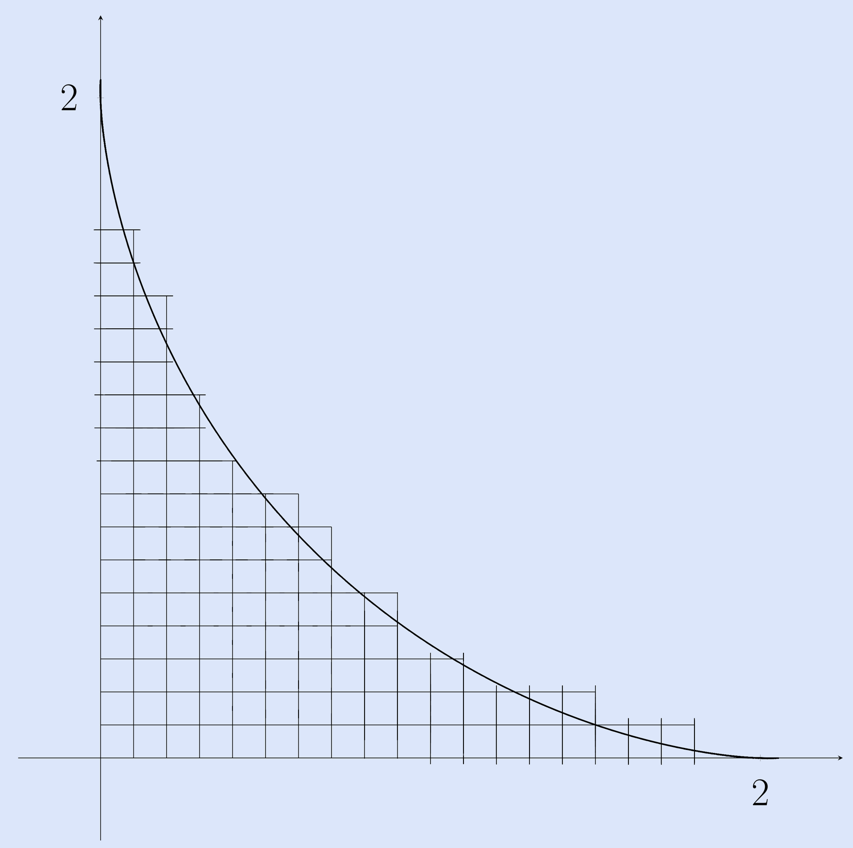


Figure 1: The limit shape Γ and a $\frac{1}{10}$-rescaled partition $\lambda \in \mathsf{Par}(100)$.

Plancherel mesure: Let $\mathsf{sh}(w)$ be the shape of w under Schensted insertion, we define

$$\mathbb{P}_n^*(\lambda) := \mathbb{P}_n(\mathsf{sh}(w) = \lambda) = \frac{f_\lambda^2}{n!}, \quad \text{for all} \quad \lambda \in \mathsf{Par}(n).$$

Vershik-Kerov-Logan-Shepp (1977): For any $\epsilon > 0$, we have

$$\mathbb{P}_n^*\left((1-\epsilon)\Gamma \subset \frac{1}{\sqrt{n}} S_\lambda \subset (1+\epsilon)\Gamma\right) \longrightarrow 1 \quad \text{as} \quad n \longrightarrow \infty.$$

Lemma 1: Let $L(\lambda)$ denote the length of the first row of the shape λ. We have

$$\mathbb{P}_n^*\left(L(\lambda) \ge \sqrt{2n}\right) \longrightarrow 1 \quad \text{as} \quad n \longrightarrow \infty.$$

Lemma 2: If $L(\lambda) \ge \sqrt{2n}$, then the shape λ has two columns of the same height.

Proof strategy

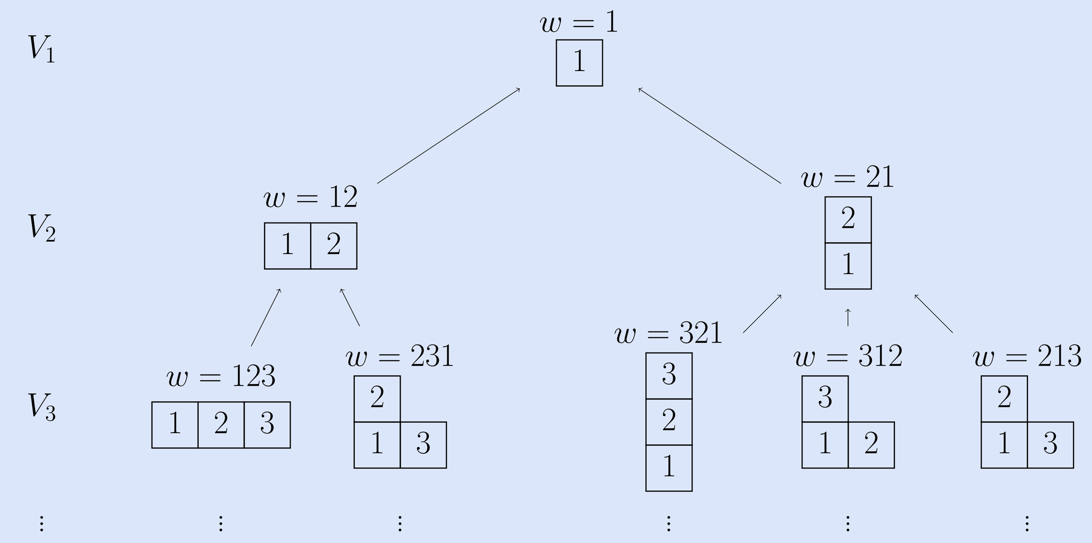


Figure 2: The rooted tree of permutations with no lateral bumps.

Flattening: Given an injective n-word $a_1a_2\ldots a_n$, its flattening is the permutation

$$\mathsf{Flat}(a_1a_2\ldots a_n) := b_1b_2\ldots b_n, \quad \text{where } b_i := \left|\{j \in [n] \mid a_j \le a_i\}\right|.$$

S-map: We define the map $\varphi_n : S_{n+1} \to S_n$ by

$$\varphi_n(w_1w_2\ldots w_nw_{n+1}) := \mathsf{Flat}(w_1w_2\ldots w_n).$$

S-inverse: Let $k^* := k + \frac{1}{2}$. The inverses of $v \in S_n$ via φ_n are

$$\mathsf{Flat}(v\,0^*), \mathsf{Flat}(v\,1^*), \ldots, \mathsf{Flat}(v\,n^*).$$

V-map: Since $\varphi_n(V_{n+1}) \subset V_n$, we can define

$$\psi_n : V_{n+1} \to V_n, \quad \text{as} \quad \varphi_n|_{V_{n+1}}.$$

Lemma 3: If the shape $\mathsf{sh}(v)$ of $v \in V_n$ has two columns of the same height, then

$$\left|\psi_n^{-1}(v)\right| \le n, \quad \text{i.e., } v \text{ has at most } n \text{ children.}$$

Lemma 4: For n large enough, we have

$$|V_{n+1}| \le n \cdot |V_n \setminus U_n| + (n+1)\cdot|U_n| \le \left(n + \frac{1}{2}\right)|V_n|.$$

Acknowledgments

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References

[1] Stelzer-Yong (2024)
[2] Vershik-Kerov (1977) and Logan-Shepp (1977)
[3] Romik (2015)
[4] Stanley (1999) and Fulton (1997)