MATH586: Schur Polynomials

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Spring 2026

1 Introduction

Definition 1. We recall the definition of **Schur polynomials** in n variables corresponding to a partition λ to be

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T.$$

Note that $\mathrm{SSYT}(\lambda)$ denotes the set of all semi-standard Young tableaux of shape λ (with boxes filled with entries in $[n]$), and $x^T = \prod_{j=1}^n x_j^{\#\{j\text{'s in } T\}}$

Schur polynomials are an integral part of algebraic combinatorics. They connect fields like representation theory and algebraic geometry to simple combinatorial boxes and numbers. The irreducibles of $GL_n(\mathbb{C})$ are indexed by partitions λ and have character s_λ. This means that decomposing polynomials in terms of Schur polynomials can help decompose representations of $GL_n(\mathbb{C})$ into irreducibles. The product of Schur polynomials is equivalent to multiplying Schubert classes. Importantly, Schur polynomials are also symmetric ($s_\lambda \in \Lambda$) and they form a basis (over $\mathbb{Z}$) of Λ. Specifically, s_λ for $\lambda \vdash k$ is a basis of $\Lambda^{(k)}$.

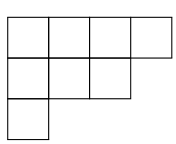
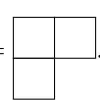
In class, we gave a combinatorial proof that s_λ is symmetric using the *Bender–Knuth involution*. The key idea is as follows: for each simple transposition $s_i = (i\ \ i{+}1)$ and each semi-standard Young tableau T of shape λ, one can construct a tableau $T' \in \mathrm{SSYT}(\lambda)$ satisfying $x^{T'} = s_i(x^T)$. The map $T \mapsto T'$ swaps the entries i and $i+1$ and then reorders them within each row to restore the semi-standard condition. This map is an involution, and it follows that s_λ is invariant under every adjacent transposition, and thus, symmetric.

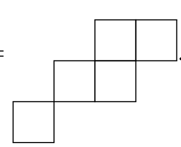
In these notes, we give an algebraic proof of a stronger statement: *the skew Schur polynomial $s_{\lambda/\mu}$ is symmetric*. We begin by defining skew partitions.

2 Skew Partitions and Skew Schur Polynomials

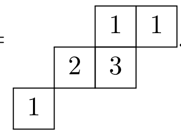
Definition 2. Let partitions λ and μ be such that $\mu \subseteq \lambda$, where μ contained in λ means the Young diagram of μ is completely covered by the Young diagram of λ when top-left justified (equivalently, $\mu_i \leq \lambda_i$ for all i). The **skew shape** λ/μ is obtained by removing the boxes of μ from λ.

A **semi-standard Young tableau of shape** λ/μ is a filling of the boxes of λ/μ with positive integers such that entries are weakly increasing along each row (left to right) and strictly increasing down each column.

Example 3. Let $\lambda =$  and $\mu =$ . Since $\mu \subseteq \lambda$, then

$\lambda/\mu =$ .

An example of a semi-standard Young tableau of shape λ/μ is

$T_{\lambda/\mu} =$ 

In this example, $x^{T_{\lambda/\mu}} = x_1^3 x_2 x_3$.

Definition 4. The **skew Schur polynomial** corresponding to a skew shape λ/μ is

$$s_{\lambda/\mu}(x_1, \dots, x_n) = \sum_{T \in \mathrm{SSYT}(\lambda/\mu)} x^T.$$

Theorem 5. *For any partitions $\mu \subseteq \lambda$, the skew Schur polynomial $s_{\lambda/\mu}$ is symmetric.*

This can be shown in a similar method as the proof of s_λ (using the Bender-Knuth involution). However, we argue that claim using an algebraic method.

3 The Operators U_k and u_i

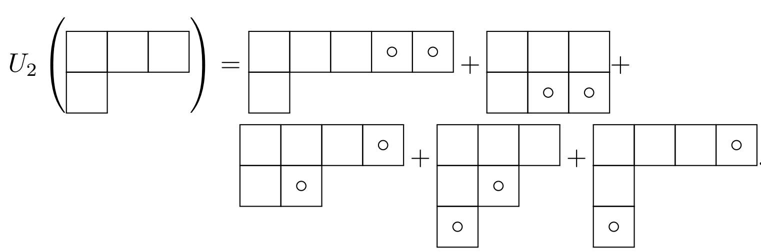
First, we define the $\mathbb{Z}$-module, $\mathbb{Z}[\mathbb{Y}]$ as the set of formal linear combinations of partition shapes $\lambda \vdash n$ for some $n \in \mathbb{N}$.

Definition 6. For $k \in \mathbb{N}$, define the operator $U_k : \mathbb{Z}[\mathbb{Y}] \to \mathbb{Z}[\mathbb{Y}]$ by

$$U_k(\mu) = \sum_{\substack{\mu \subseteq \lambda \\ |\lambda/\mu| = k \\ (\lambda'_i - \mu'_i) \leq 1 \ \forall i}} \lambda \ ,$$

where the sum ranges over all partitions λ obtained from μ by adding a *horizontal strip* of size k (adding k boxes with at most one new box per column). Simply put, $U_k(\lambda)$ is the sum of all possibly obtainable partition shapes by adding a box in k distinct columns.

Example 7. There are five possible tableau that can be obtained by adding 2 boxes in the form of horizontal strips to $\lambda = (3, 1)$. The boxes with circles denote the horizontal strips.

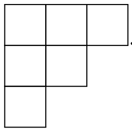


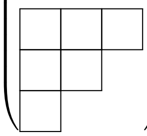
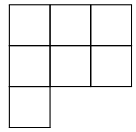
We can also express U_k in a different way.

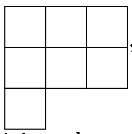
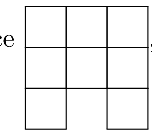
Definition 8. For each $i \geq 1$, define $u_i : \mathbb{Z}[\mathbb{Y}] \to \mathbb{Z}[\mathbb{Y}]$ by

$$u_i(\lambda) = \begin{cases} (\lambda'_1, \dots, \lambda'_i + 1, \dots, \lambda'_{\ell(\lambda')+1})' & \text{if the result is a valid partition,} \\ 0 & \text{otherwise.} \end{cases}$$

In words, u_i adds a single box in column i of λ, if doing so yields a valid partition, and is 0 otherwise.

Example 9. Let $\lambda =$ . Adding a box in column 3 is valid, so

$$u_3(\lambda) = u_3\left(\right.$$  $$\left. \right) =$$  .

However, for $\mu =$ , $u_3(\mu) = 0$, since adding a box in column 3 would produce , which is not a valid partition shape.

Using the operators u_i, we can express U_k as a sum over products:

$$U_k = \sum_{i_1 > i_2 > \cdots > i_k \geq 1} u_{i_1}\, u_{i_2} \cdots u_{i_k} \quad .$$

4 From Operators to Skew Schur Polynomials

Next, we define $\mathbb{Z}[[x]][U_k]$ as the ring of formal power series in $\mathbb{Z}$ adjoined with all U_k's. In other words, it is the polynomial ring in U_k's with coefficients in $\mathbb{Z}[[x]]$. Then, define

$$A(x) := \sum_{k=0}^{\infty} U_k\, x^k = \prod_{i=\infty}^{1} (1 + x\, u_i) = \cdots (1 + x\, u_3)(1 + x\, u_2)(1 + x\, u_1) \quad .$$

This formula indeed holds because the left side represents all ways of adding a single valid horizontal strip of arbitrary length, and the right side represents all ways of adding a single box in arbitrarily many columns.

Remark 10. *Let μ be a partition and $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a composition. Then the coefficient of λ in $U_{\alpha_m} U_{\alpha_{m-1}} \cdots U_{\alpha_1}(\mu)$ counts the number of semi-standard Young tableaux of shape λ/μ with content α.*

To see this, note that each U_{α_i} adds a horizontal strip of size α_i. We label the boxes added by U_{α_i} with the entry i. The horizontal strip condition ensures that no column receives more than one box per application, so entries increase strictly down columns. Since each successive U_{α_i} acts on a partition that already contains all previously added boxes, and the new boxes are placed in weakly higher-indexed positions within each row, the entries are weakly increasing along rows. Thus the resulting filling is indeed a semi-standard Young tableau.

Proposition 11. *For some partition μ,*

$$\sum_{\lambda \supseteq \mu} s_{\lambda/\mu}(x_1, x_2, \ldots) \cdot \lambda \;=\; \cdots A(x_2)\, A(x_1)\, \mu \quad .$$

5 Commutativity and Symmetry

We now complete the algebraic proof of Theorem 5 using the following theorem.

Theorem 12. *The generating series commute. In other words,*

$$A(x)A(y) = A(y)A(x) \quad .$$

Proof. Since $A(x) = \cdots (1 + xu_3)(1 + xu_2)(1 + xu_1)$ is a product of factors indexed by column position, it suffices to prove the following two relations:

(i) $[\,(1+xu_{i+1})(1+xu_i),\ (1+yu_{i+1})(1+yu_i)\,] = 0$ for all i.

(ii) $u_i\,u_j = u_j\,u_i$ whenever $|i-j|>1$.

Proof of (i). We expand the Lie bracket. The first term is

$$\begin{aligned} P = 1 &+ (x+y)u_{i+1} + (x+y)u_i + xy\,(u_{i+1}^2 + u_i^2 + u_i u_{i+1}) \\ &+ (x^2+xy+y^2)\,u_{i+1}u_i + xy^2\,u_{i+1}^2 u_i + x^2y\,u_{i+1}u_iu_{i+1} \\ &+ x^2y\,u_{i+1}u_i^2 + xy^2\,u_iu_{i+1}u_i + x^2y^2\,u_{i+1}u_iu_{i+1}u_i \quad . \end{aligned}$$

Let Q denote the same expansion with x and y swapped. The terms symmetric in x, y cancel in $P-Q$, leaving

$$P - Q = xy(x-y)\big(u_{i+1}u_iu_{i+1} + u_{i+1}u_i^2 - u_{i+1}^2u_i - u_iu_{i+1}u_i\big) \quad .$$

Observe that if the following two relations hold, $P-Q$ vanishes to 0:

$$u_{i+1}^2u_i = u_{i+1}u_iu_{i+1} \tag{1}$$

$$u_{i+1}u_i^2 = u_iu_{i+1}u_i \tag{2}$$

These relations can be combinatorially observed by noting that the output of some $u_j(\lambda)$ is completely dependent on λ'_{j-1} (see Prop. 3.5 of [LS19]).

Proof of (ii). Suppose $|i-j|>1$. The operator u_j adds a box at the bottom of column j, and whether this is valid depends only on the column heights λ'_{j-1} and λ'_j. Since u_i modifies only column i, which is not adjacent to column j, the validity and outcome of u_j is unaffected by whether u_i has been applied first, and vice versa. Moreover, $i \neq j$ means the two operators add boxes in distinct columns. Therefore $u_iu_j = u_ju_i$. ☐

To see why Theorem 12 implies the symmetry of $s_{\lambda/\mu}$, consider any two adjacent variables x_i and x_{i+1}. Using commutativity of $A(x_i)$ and $A(x_{i+1})$, we can swap them in the infinite product:

$$\cdots A(x_{i+1})\,A(x_i)\cdots \;=\; \cdots A(x_i)\,A(x_{i+1})\cdots \quad .$$

Since the coefficient of λ on the left-hand side is $s_{\lambda/\mu}(x_1,\dots,x_i,x_{i+1},\dots)$ and the coefficient of λ on the right-hand side is $s_{\lambda/\mu}(x_1,\dots,x_{i+1},x_i,\dots)$, we conclude that $s_{\lambda/\mu}$ is invariant under every adjacent transposition. Because the adjacent transpositions generate the full symmetric group, it follows that $s_{\lambda/\mu}$ is symmetric.

6 Exercises and Further Discussion

6.1 Relations on simple box-adding operators

Problem 13. What are all the relations on words in the generators $u_1, u_2, \dots$?

This had been a longstanding question since the 1990s, posed by Fomin (see [Fom95]). The full set of relations is made up of the three relations used in the proof of Theorem 12 and an additional relation:

$$u_{i+1}u_{i+2}u_{i+1}u_i = u_{i+1}u_{i+2}u_iu_{i+1},$$

as described in [LS19].

6.2 Connection to Representation Theory

Definition 14. Let $\lambda \vdash k$ be a partition and $T \in \mathrm{SSYT}(\lambda)$. Label the fillings in column i as $i_1, i_2, \dots, i_{\#(i)}$. Then denote

$$P_T := \prod_{i=1}^{k} \begin{vmatrix} z_{i_1 1} & z_{i_1 2} & \cdots & z_{i_1 \#(i)} \\ z_{i_2 1} & z_{i_2 2} & & \vdots \\ \vdots & & \ddots & \\ z_{i_{\#(i)} 1} & \cdots & & z_{i_{\#(i)} \#(i)} \end{vmatrix} .$$

The **Schur module** V_λ, with respect to a partition λ, is then defined as

$$V_\lambda := \mathrm{span}_{\mathbb{C}} \{P_T : T \in \mathrm{SSYT}(\lambda)\} .$$

Problem 15. *Show that $GL_n \curvearrowright V_\lambda$ is a valid action and produces a GL_n-representation. It suffices to show this for the elementary matrices. As a GL_n-representation, the entries of $T \in \mathrm{SSYT}(\lambda)$ are restricted to $\{1, \ldots, n\}$.*

Problem 16. *One can check that the character of V_λ is exactly $s_\lambda(x_1, \ldots, x_n)$. Find the highest weight vector of V_λ. Recall that the highest weight vector is the unique (up to scaling) vector that is fixed under the Borel subgroup. Then, prove that V_λ is irreducible as a GL_n-representation. Recall that a GL_n-representation, $\rho : GL_n \to GL(V)$, is polynomial if the entries $\rho(g)$ are polynomials of the entries of g for all $g \in GL_n$.*

Problem 17. *It can be seen that for all partitions λ, there exists an irreducible (polynomial) V_λ. Are there other polynomial irreducibles that do not correspond to a partition?*

Just as we extended Schur polynomials to skew Schur polynomials, one can define *skew Schur modules* $V_{\lambda/\mu}$. These decompose as

$$V_{\lambda/\mu} \;\cong\; \bigoplus_{\nu} c^{\lambda}_{\mu,\nu}\, V_\nu \;\; ,$$

where $c^{\lambda}_{\mu,\nu}$ are the *Littlewood–Richardson coefficients*. This decomposition is a direct consequence of the corresponding identity for skew Schur polynomials:

$$s_{\lambda/\mu}(x_1, \ldots, x_n) \;=\; \sum_{\nu} c^{\lambda}_{\mu,\nu}\, s_\nu(x_1, \ldots, x_n) \;\; .$$

References

[Fom95] Sergey Fomin. Schur Operators and Knuth Correspondences. *Journal of Combinatorial Theory, Series A*, 72(2):277–292, March 1995. doi:10.1016/0097-3165(95)90065-9.

[LS19] Ricky Ini Liu and Christian Smith. The Algebra of Schur Operators. July 2019. doi:10.48550/arXiv.1907.05824.