MATH586: The Cauchy Identity

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Spring 2026

1 Introduction

The Cauchy Identity of Schur Polynomials is an important result in algebraic combinatorics and the theory of symmetric functions. It provides a fundamental connection between the ring of symmetric functions and combinatorial objects such as Young tableaux.

We begin by recalling that the *Schur polynomial* $S_\lambda(X)$ associated to a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ in the variables $X = (x_1, x_2, \ldots, x_n)$ is defined by

$$S_\lambda(x_1, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T,$$

where the sum ranges over all semistandard Young tableaux T of shape λ with entries in $\{1, 2, \ldots, n\}$.

Theorem 1 (Cauchy Identity)**.** *The Cauchy Identity reads*

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda S_\lambda(X) S_\lambda(Y), \tag{1}$$

where $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_m)$ are two sets of indeterminates.

2 Proof of Theorem 1

The Cauchy Identity admits a well-known bijective proof via the Robinson–Schensted–Knuth (RSK) correspondence. In these notes we give an alternative algebraic proof that instead relies on commutation relations of operators acting on the space of partitions.

2.1 The operators U_k, u_i, D_k, and d_j

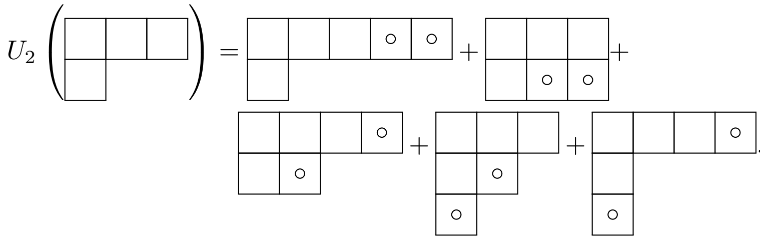
First, we define the $\mathbb{Z}$-module, $\mathbb{Z}[\mathbb{Y}]$ as the set of formal linear combinations of partition shapes $\lambda \vdash n$ for some $n \in \mathbb{N}$.

Definition 2. For $k \in \mathbb{N}$, define the operator $U_k : \mathbb{Z}[\mathbb{Y}] \to \mathbb{Z}[\mathbb{Y}]$ by

$$U_k(\lambda) = \sum_{\substack{\lambda \subseteq \mu \\ |\mu/\lambda| = k \\ (\mu'_i - \lambda'_i) \leq 1 \forall i}} \mu,$$

where the sum ranges over all partitions μ obtained from λ by adding a *horizontal strip* of size k, i.e. by adding k boxes with at most one new box per column.

Example 3. There are five possible partitions that can be obtained by adding 2 boxes in the form of horizontal strips to $\lambda = (3,1)$. The boxes with circles denote the horizontal strips.

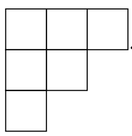


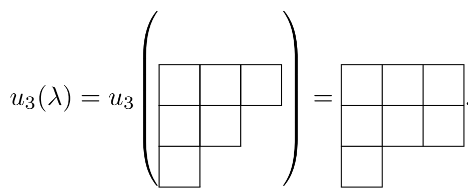
We can also express U_k in a different way.

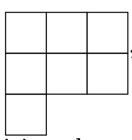
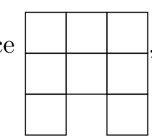
Definition 4. For each $i \geq 1$, define $u_i : \mathbb{Z}[\mathbb{Y}] \to \mathbb{Z}[\mathbb{Y}]$ by

$$u_i(\lambda) = \begin{cases} (\lambda'_1, \ldots, \lambda'_i + 1, \ldots, \lambda'_{\ell(\lambda')+1})' & \text{if the result is a valid partition,} \\ 0 & \text{otherwise.} \end{cases}$$

In words, u_i adds a single box in column i of λ, if doing so yields a valid partition, and is 0 otherwise.

Example 5. Let $\lambda =$ . Adding a box in column 3 is valid, so



However, for $\mu =$ , $u_3(\mu) = 0$, since adding a box in column 3 would produce , which is not a valid partition shape.

Using the operators u_i, we can express U_k as a sum over products:

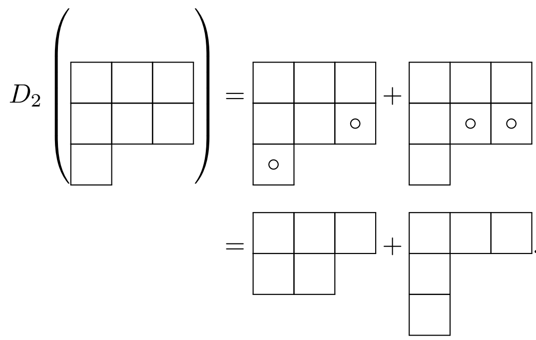
$$U_k = \sum_{i_1 > i_2 > \cdots > i_k \geq 1} u_{i_1} u_{i_2} \cdots u_{i_k}.$$

Definition 6. For $k \in \mathbb{N}$, define the operator $D_k : \mathbb{Z}[\mathbb{Y}] \to \mathbb{Z}[\mathbb{Y}]$ by

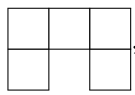
$$D_k(\lambda) = \sum_{\substack{\mu \subseteq \lambda \\ |\lambda/\mu| = k \\ (\lambda'_i - \mu'_i) \leq 1 \forall i}} \mu,$$

where the sum ranges over all partitions μ obtained from λ by removing a *horizontal strip* of size k, i.e. by removing k boxes with at most one removed box per column. Note that the resulting shape μ must itself be a valid partition, meaning the row lengths are weakly decreasing: $\mu_1 \geq \mu_2 \geq \cdots \geq 0$.

Example 7. There are three partitions that can be obtained by removing 2 boxes in the form of horizontal strips from $\lambda = (3,3,1)$. The boxes with circles denote the removed horizontal strips.



Note that removing boxes from columns 1 and 2 would yield the following shape



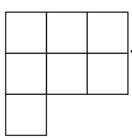
which would not be valid.

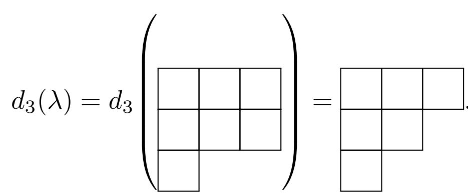
Similarly, we can also represent D_k as a sum over products of simpler operators.

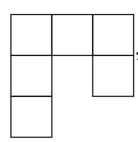
Definition 8. For each $j \geq 1$, define $d_j : \mathbb{Z}[\mathbb{Y}] \to \mathbb{Z}[\mathbb{Y}]$ by

$$d_j(\lambda) = \begin{cases} (\lambda'_1, \ldots, \lambda'_j - 1, \ldots, \lambda'_{\ell(\lambda')})' & \text{if the result is a valid partition,} \\ 0 & \text{otherwise.} \end{cases}$$

In words, d_j removes a single box from column j of λ, if doing so yields a valid partition, and is 0 otherwise.

Example 9. Let $\lambda =$ . Removing a box from column 3 is valid, so



However, $d_2(\lambda) = 0$, since removing a box from column 2 would produce , which is not a valid partition shape.

Using the operators d_j, we can express D_k as a sum over products:

$$D_k = \sum_{j_1 > j_2 > \cdots > j_k \geq 1} d_{j_1} d_{j_2} \cdots d_{j_k}.$$

2.2 The Commutation Lemma

Lemma 10. *If $i \neq j$, $u_i d_j = d_j u_i$.*

Proof. Adding a box in column i and removing a box in column $j \neq i$ are independent operations that do not interfere with each other. □

Lemma 11. $u_i d_i = d_{i+1} u_{i+1}$.

Proof. The left-hand side first attempts to remove a box from column i with d_i. If the result of the removal is a valid shape, then, by adding the box back with u_i, the composition is the identity. The result of the removal d_i is not a valid shape exactly when the number of boxes in column $i+1$ is equal to the number of boxes in column i. In this case, the composition is 0. The right-hand side first attempts to add a box to column $i+1$ with u_{i+1}. If the result of the addition is a valid shape, then by removing the box with d_{i+1}, the composition is the identity. The result of the addition is not a valid shape exactly when the number of boxes in column $i+1$ is equal to the number of boxes in column i. In this case, the composition is 0. Thus, we see that both compositions are the same. □

Lemma 12. $d_1 u_1 = \mathrm{Id}$.

Proof. u_1 adds a box in column 1 (appends a new row of length 1), and d_1 removes a box from column 1 (removes the new row of length 1), so the composition is the identity. □

Lemma 13. *In any associative algebra, if a and b are elements such that both $(1-ba)^{-1}$ and $(1-ab)^{-1}$ exist, then*

$$(1+a)(1-ba)^{-1}(1+b) = (1+b)(1-ab)^{-1}(1+a).$$

Proof. Expand both sides using the geometric series $(1-ba)^{-1} = \sum_{n\geq 0}(ba)^n$ and $(1-ab)^{-1} = \sum_{n\geq 0}(ab)^n$. For the left-hand side,

$$\begin{aligned}(1+a)(1-ba)^{-1}(1+b) &= \sum_{n\geq 0}(1+a)(ba)^n(1+b) \\ &= \sum_{n\geq 0}\big[(ba)^n + a(ba)^n + (ba)^n b + a(ba)^n b\big].\end{aligned}$$

Using the identities $a(ba)^n = (ab)^n a$, $(ba)^n b = b(ab)^n$, and $a(ba)^n b = (ab)^{n+1}$, this becomes

$$\mathrm{LHS} = \sum_{n\geq 0}\big[(ba)^n + (ab)^n a + b(ab)^n + (ab)^{n+1}\big].$$

For the right-hand side,

$$\begin{aligned}(1+b)(1-ab)^{-1}(1+a) &= \sum_{n\geq 0}(1+b)(ab)^n(1+a) \\ &= \sum_{n\geq 0}\big[(ab)^n + b(ab)^n + (ab)^n a + b(ab)^n a\big].\end{aligned}$$

Using $b(ab)^n a = (ba)^{n+1}$, this becomes

$$\mathrm{RHS} = \sum_{n\geq 0}\big[(ab)^n + b(ab)^n + (ab)^n a + (ba)^{n+1}\big].$$

Taking the difference:

$$\begin{aligned}\mathrm{LHS} - \mathrm{RHS} &= \sum_{n\geq 0}\Big(\big[(ba)^n + (ab)^{n+1}\big] - \big[(ab)^n + (ba)^{n+1}\big]\Big) \\ &= \sum_{n\geq 0}(ba)^n + \sum_{n\geq 1}(ab)^n - \sum_{n\geq 0}(ab)^n - \sum_{n\geq 1}(ba)^n \\ &= (ba)^0 - (ab)^0 \\ &= 1 - 1 \\ &= 0.\end{aligned}$$

□

We now express the operators as generating functions. Define the formal power series

$$A(x) := \sum_{k=0}^{\infty} U_k x^k,$$

which encodes all horizontal-strip additions, weighted by the number of boxes added. Using the decomposition $U_k = \sum_{i_1 > \cdots > i_k \geq 1} u_{i_1} \cdots u_{i_k}$, this generating function factors as an ordered product of binomial terms:

$$A(x) = \prod_{i=\infty}^{1} (1 + xu_i) = \cdots (1 + xu_3)(1 + xu_2)(1 + xu_1).$$

Each factor $(1+xu_i)$ either does nothing (the 1 term) or adds a box in column i (the xu_i term). The product is ordered with indices decreasing from left to right, which matches the strict ordering $i_1 > \cdots > i_k$ in the definition of U_k and ensures we obtain all horizontal strips.

Similarly, define

$$B(y) := \sum_{k=0}^{\infty} D_k y^k,$$

which encodes all horizontal-strip removals. This factors as

$$B(y) = \prod_{j=1}^{\infty} (1 + yd_j) = (1 + yd_1)(1 + yd_2)(1 + yd_3) \cdots ,$$

with indices increasing from left to right.

Lemma 14 (Commutation Lemma)**.** $B(y)A(x) = A(x)B(y)\frac{1}{1 - xy}$.

Proof. Let λ be an arbitrary partition. Then there exists $N \in \mathbb{N}$ such that

$$(1 + xu_N)\lambda = \lambda \quad \text{and} \quad (1 + yd_N)\lambda = \lambda,$$

since attempting to add or remove a box in the N-th column of λ when λ has fewer than N columns results in 0. Thus, it suffices to prove the finite version of the Cauchy Identity:

$$(1 + yd_1) \cdots (1 + yd_N)(1 + xu_N) \cdots (1 + xu_1) = (1 + xu_N) \cdots (1 + xu_1)(1 + yd_1) \cdots (1 + yd_N)\frac{1}{1 - xy}. \quad (2)$$

Since $\frac{1}{1-xy}$ is a scalar, it commutes with all operators, and the right-hand side of (2) becomes

$$(1 + xu_N) \cdots (1 + xu_2) \underbrace{(1 + xu_1)\frac{1}{1 - xy}(1 + yd_1)}(1 + yd_2) \cdots (1 + yd_N). \quad (3)$$

Observe that the middle three terms of (3) can be evaluated as follows.

$$\begin{aligned}(1 + xu_1)\frac{1}{1 - xy}(1 + yd_1) &= (1 + xu_1)\frac{1}{1 - yd_1 \cdot xu_1}(1 + yd_1) && \text{By Lemma 12} \\ &= (1 + yd_1)\frac{1}{1 - xu_1 \cdot yd_1}(1 + xu_1) && \text{By Lemma 13} \\ &= (1 + yd_1)\frac{1}{1 - yd_2 \cdot xu_2}(1 + xu_1) && \text{By Lemma 11}\end{aligned}$$

By Lemma 10, u_i and d_j commute for $i \neq j$, so we can pass $(1+xu_1)$ to the right past $(1+yd_2) \cdots (1+yd_N)$ and pass $(1 + yd_1)$ to the left past $(1 + xu_N) \cdots (1 + ux_2)$, yielding

$$(1 + yd_1)(1 + xu_N) \cdots (1 + xu_2) \underbrace{(1 + xu_2)\frac{1}{1 - yd_2 \cdot xu_2}(1 + yd_2)}(1 + yd_3) \cdots (1 + yd_N)(1 + xu_1).$$

We now repeat the argument. After the i-th step, the expression has the form

$$(1+yd_1)\cdots(1+yd_i)(1+xu_N)\cdots(1+xu_{i+1})\frac{1}{1-yd_{i+1}\cdot xu_{i+1}}(1+yd_{i+1})\cdots(1+yd_N)(1+xu_i)\cdots(1+xu_1).$$

After N steps, we obtain

$$(1+yd_1)\cdots(1+yd_N)\frac{1}{1-yd_{N+1}\cdot xu_{N+1}}(1+xu_N)\cdots(1+xu_1).$$

Since there is no box to add or remove in column $N+1$, we have that $\frac{1}{1-xyd_{N+1}u_{N+1}} = \mathrm{Id}$. Therefore

$$A(x)B(y)\frac{1}{1-xy} = (1+yd_1)\cdots(1+yd_N)(1+xu_N)\cdots(1+xu_1) = B(y)A(x). \qquad \square$$

2.3 How the Commutation Lemma Implies the Cauchy Identity

We first establish the connection between the operators $A(x)$, $B(y)$ and Schur polynomials.

Lemma 15.

$$A(x_n)\cdots A(x_1)(\varnothing) = \sum_{\lambda} S_\lambda(x_1,\dots,x_n)\lambda,$$

where $\varnothing$ denotes the empty partition.

Proof. Each application of $A(x_i)$ adds a horizontal strip, weighted by x_i^k where k is the number of boxes added. A sequence of horizontal strip additions

$$\varnothing = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \cdots \subseteq \lambda^{(n)} = \lambda$$

corresponds to a semistandard Young tableau T of shape λ with entries in $\{1,\dots,n\}$. The horizontal strip condition ensures that no two boxes with the same entry appear in the same column, which is precisely the column-strict condition for semistandard tableaux. The total weight of such a sequence is $x_1^{|\lambda^{(1)}|}x_2^{|\lambda^{(2)}/\lambda^{(1)}|}\cdots x_n^{|\lambda/\lambda^{(n-1)}|} = x^T$. Summing over all such sequences for a given λ yields $S_\lambda(x_1,\dots,x_n)$. $\square$

By the same reasoning applied in reverse, successive horizontal strip removals correspond to reading a semistandard Young tableau in reverse.

Lemma 16. *For any partition λ,*

$$[\varnothing]B(y_m)\cdots B(y_1)(\lambda) = S_\lambda(y_1,\dots,y_m),$$

where $[\varnothing]$ denotes the coefficient of the empty partition $\varnothing$.

Proof. Each application of $B(y_j)$ removes a horizontal strip, weighted by y_j^k where k is the number of boxes removed. A sequence of removals

$$\lambda = \mu^{(0)} \supseteq \mu^{(1)} \supseteq \cdots \supseteq \mu^{(m)} = \varnothing$$

corresponds to a semistandard Young tableau of shape λ with entries in $\{1,\dots,m\}$. The total weight of such a sequence is $y_1^{|\mu^{(0)}/\mu^{(1)}|}\cdots y_m^{|\mu^{(m-1)}|} = y^T$. Summing over all such sequences gives $S_\lambda(y_1,\dots,y_m)$. $\square$

We can now complete the proof of the Cauchy Identity. By repeated application of Lemma 14, we commute all the B operators past all the A operators:

$$B(y_m)\cdots B(y_1)A(x_n)\cdots A(x_1) = A(x_n)\cdots A(x_1)B(y_m)\cdots B(y_1)\prod_{i,j}\frac{1}{1-x_iy_j}.$$

Apply both sides to the empty partition $\varnothing$ and extract the coefficient of $\varnothing$.

Left-hand side:

$$\begin{aligned}
[\varnothing]B(y_m)\cdots B(y_1)A(x_n)\cdots A(x_1)(\varnothing) &= \sum_\lambda S_\lambda(X)[\varnothing]B(y_m)\cdots B(y_1)(\lambda) && \text{By Lemma 15}\\
&= \sum_\lambda S_\lambda(X)S_\lambda(Y) && \text{By Lemma 16}
\end{aligned}$$

Right-hand side: Since there are no boxes to remove from the empty partition, $B(y_m)\cdots B(y_1)(\varnothing) = \varnothing$. Then, by Lemma 15, $A(x_n)\cdots A(x_1)(\varnothing) = \sum_\lambda S_\lambda(X)\lambda$. Extracting the coefficient of $\varnothing$ gives $S_\varnothing(X) = 1$. The right-hand side evaluates to

$$1 \cdot \prod_{i,j} \frac{1}{1 - x_i y_j} = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

Equating the two sides gives the Cauchy Identity:

$$\sum_\lambda S_\lambda(X)S_\lambda(Y) = \prod_{i,j} \frac{1}{1 - x_i y_j}. \quad \square$$

3 Consequences of the Cauchy Identity

In this section, we derive several corollaries of the Cauchy Identity. First, recall that

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda m_\lambda(x)h_\lambda(y), \tag{4}$$

where m_λ is the monomial symmetric function, $h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_\ell}$, and h_k is the sum of all monomials of degree k. Using (1) and (4), we obtain the identity

$$\sum_\lambda S_\lambda(x)S_\lambda(y) = \sum_\lambda m_\lambda(x)h_\lambda(y). \tag{5}$$

Corollary 17. *The Schur polynomials $\{S_\lambda\}$ form an orthonormal basis of the ring of symmetric functions Λ:*

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}.$$

Proof. See Corollary 7.12.2 of [Sta99]. $\square$

Corollary 18. *For all partitions $\mu, \nu \vdash n$, let $N_{\mu\nu}$ denote the number of contingency tables (matrices with nonnegative integer entries) with row sums μ and column sums ν. Then*

$$\sum_{\lambda \vdash n} K_{\lambda\mu}K_{\lambda\nu} = N_{\mu\nu},$$

where $K_{\lambda\mu}$ is the Kostka coefficient, defined as the number of semistandard Young tableaux of shape λ and content μ.

Proof. We extract the coefficient of $x^\mu y^\nu$ from both sides of (5). On the right-hand side, $[x^\mu]m_\lambda(x) = 1$ if $\lambda = \mu$ and 0 otherwise, so only the $\mu = \lambda$ term survives. For the y-factor, $[y^\nu]h_\lambda(y)$ counts the number of matrices with nonnegative integer entries whose row sums are determined by λ and whose column sums are ν, so $[y^\nu]h_\mu(y) = N_{\mu\nu}$. Thus,

$$[x^\mu y^\nu]\sum_\lambda m_\lambda(x)h_\lambda(y) = N_{\mu\nu}.$$

On the left-hand side, the Kostka coefficient gives $[x^\mu]S_\lambda(x) = K_{\lambda\mu}$, and similarly, $[y^\nu]S_\lambda(y) = K_{\lambda\nu}$. Thus,

$$[x^\mu y^\nu]\sum_{\lambda \vdash n} S_\lambda(x)S_\lambda(y) = \sum_{\lambda \vdash n} K_{\lambda\mu}K_{\lambda\nu}.$$

$\square$

Corollary 19. *It holds that*

$$h_\mu = \sum_\lambda K_{\lambda\mu} S_\lambda.$$

Proof. Extract the coefficient of x^μ from both sides of (5). On the right-hand side, $[x^\mu]m_\lambda(x) = 1$ if $\lambda = \mu$ and 0 otherwise. As only the $\lambda = \mu$ term survives, the right-hand side is just $h_\mu(y)$. On the left-hand side, $[x^\mu]S_\lambda(x) = K_{\lambda\mu}$, so we obtain $\sum_\lambda K_{\lambda\mu} S_\lambda(y)$. □

References

[Sta99] Richard P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, 1999.